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Kosterlitz-Thouless transitions on a fluctuating surface of genus zero

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We investigate the Kosterlitz-Thouless transition for hexatic order on a fluctuating spherical surface of genus zero and derive a Coulomb gas Hamiltonian to describe it. In the Coulomb gas Hamiltonian, charge densities arise from disclinations and from Gaussian curvature. There is an interaction coupling the difference between these two densities, whose strength is determined by the hexatic rigidity. We then convert it into the sine-Gordon Hamiltonian and find a linear coupling between a scalar field and the Gaussian curvature. After integrating over the shape fluctuations, we obtain the massive sine-Gordon Hamiltonian, which corresponds to a neutral Yukawa gas, and the interaction between the disclinations is screened. We find, for $K_A/\kappa \gg 1/4$, where K_A and κ are hexatic and bending rigidity, respectively, that the transition is suppressed altogether, much as the Kosterlitz-Thouless transition is suppressed in an infinite two-dimensional superconductor. If, on the other hand, $K_A/\kappa \ll 1/4$, there can be an effective transition. [S1063-651X(96)09211-2]

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Recently, the Kosterlitz-Thouless (KT) transition for hexatic order on a free fluctuating membrane has been investigated [1]. A flat rigid membrane can have quasi-long-range (QLR) hexatic order [2] at low temperature and undergo a KT disclination unbinding transition [3–5] to a disordered high-temperature phase. A fluctuating membrane can also have QLR hexatic order [6] at low temperature. At high temperature, a fluctuating membrane has no internal order and can be charaterized by a bending rigidity κ. At length scales smaller than the persistence length $\xi_p = a e^{4\pi\kappa/3T}$, where a is a molecular size and T is the temperature, the membrane looks flat; at longer length scales, it is crumpled. However, at low temperature, hexatic order stiffens the bending rigidity so that the bending rigidity approaches a constant times the hexatic rigidity K_A [7]. Thus the hexatic membrane is more rigid than a fluid membrane and it is said to be crinkled rather than crumpled. A fluctuating hexatic membrane can undergo a KT transition from the crinkled hexatic to the crumpled fluid state. For fixed large κ, there is a disclination melting to the crumpled fluid phase as temperature is increased, and at fixed K_A, there is a transition to the crumpled fluid phase as κ is decreased.

In this paper, we extend a study of the KT transitions to a fluctuating surface of genus zero. Ovrut and Thomas discussed the structure of the KT transition of a vortex-monopole Coulomb gas on a rigid sphere and show that it is the same as in the planar case, i.e., the KT transition temperature on a rigid sphere is the same as that on the Euclidean plane: $T^{\mathrm{KT}}_{\mathrm{sphere}} = T^{\mathrm{KT}}_{\mathrm{plane}} = \pi K_A/2$ [8]. We investigate the effect of thermal shape fluctuations of a genus zero surface on the KT transition in the limit $\beta\kappa \gg 1$. In this limit, we can parametrize the surface by its radius vector as a function of standard polar coordinates $\mathbf{u} = (\theta,\phi) \equiv \Omega$,

$$\mathbf{R}(\Omega) = R[1+\rho(\Omega)]\mathbf{e}_r, \tag{1}$$

where $\mathbf{e}_r$ is the radial unit vector and $\rho(\Omega)$ measures deviation from sphericity with radius R. This parametrization is a ''normal gauge.'' To make the equations simple, we map this parametrization onto the unit sphere parametrization with $R=1$. Later, when we analyze the interaction between two disclinations, we will recover this length scale. Associated with $\mathbf{R}(\Omega)$ is a metric tensor $g_{\alpha\beta}(\Omega) = \partial_\alpha \mathbf{R}(\Omega)\cdot\partial_\beta \mathbf{R}(\Omega)$ and a curvature tensor $K_{\alpha\beta}(\Omega)$ defined via $K_{\alpha\beta}(\Omega) = \mathbf{N}(\Omega)\cdot\partial_\alpha\partial_\beta\mathbf{R}(\Omega)$, where $\mathbf{N}(\Omega)$ is the local unit normal to the surface. From the curvature tensor $K_{\alpha\beta}$, the mean curvature H and the Gaussian curvature K are defined as

$$H = \frac{1}{2} g^{\alpha\beta} K_{\beta\alpha}, \quad K = \det g^{\alpha\lambda} K_{\lambda\beta}, \tag{2}$$

where $g^{\alpha\beta}$ is the inverse tensor of $g_{\alpha\beta}$ satisfying $g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta$

To describe hexatic order, we construct the tangent vectors

$$\mathbf{t}_\theta = \partial_\theta \mathbf{R}, \quad \mathbf{t}_\phi = \partial_\phi \mathbf{R}, \tag{3}$$

where $\partial_\alpha = \partial/\partial u^\alpha$ and $\mathbf{u} = (\theta,\phi)$, and introduce orthonormal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ at each point on the surface. Then $\mathbf{e}_1(\Omega)\cdot\mathbf{t}_\theta(\Omega) = \cos\Theta(\Omega)$ defines a local bond angle $\Theta(\Omega)$. Hexatic order is then described by the local bond order parameter $\mathbf{m}(\Omega) = \cos\Theta\mathbf{e}_1 + \sin\Theta\mathbf{e}_2$, where $\Theta(\Omega)$ has sixfold symmetry. Note that since $\Theta(\Omega)$ depends on the choice of orthonormal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, any spatial derivatives for $\mathbf{m}$ must be covariant derivatives.

In the continuum elastic theory, it is now well established that the long-wavelength properties of a fluctuating membrane are described by the Helfrich-Canham Hamiltonian $\mathcal{H}_{\mathrm{HC}}$ [9] and the hexatic free energy $\mathcal{H}_A$ [6]. The Helfrich-Canham Hamiltonian can be expressed as a sum of three terms

$$\mathcal{H}_{\mathrm{HC}} = \mathcal{H}_\kappa + \mathcal{H}_G + \mathcal{H}_\sigma. \tag{4}$$

The first term is the mean curvature energy

$$\mathcal{H}_\kappa=\frac{1}{2}\kappa\int d^2u\sqrt{g}(2H-2H_0)^2$$
$$=\frac{1}{2}\kappa\int d\Omega[(\nabla^2+2)\rho]^2+O(\rho^3), \tag{5}$$

where $g=\det g_{\alpha\beta}$, $2H=K^\alpha_\alpha$ is twice the mean curvature, H_0 is the spontaneous mean curvature, which is equal to 1 for the sphere, and the second form is valid for the normal gauge. The second term is the Gaussian curvature energy

$$\mathcal{H}_G=\frac{1}{2}\kappa_G\int d^2u\sqrt{g}K, \tag{6}$$

where $K=\det K^\alpha_\beta$ is the Gaussian curvature. This term is a topological invariant depending only on the genus of the surfaces due to the Gauss-Bonnet theorem

$$\int d^2u\sqrt{g}K=4\pi(1-\eta)=2\pi\chi, \tag{7}$$

where η is the number of handles and $\chi=2(1-\eta)$ is the Euler characteristic. Since we will consider surfaces of fixed genus, we will drop this term. Finally,

$$\mathcal{H}_\sigma=\sigma\int d^2u\sqrt{g} \tag{8}$$

is the surface tension energy. We are mostly interested in free membranes for which the renormalized surface tension obtained by differentiating the total free energy $\mathcal{F}$ with respect to the total surface area $\mathcal{A}$ ($\sigma_R=\partial\mathcal{F}/\partial\mathcal{A}$) is zero. Since there are entropic contributions to σ_R as well as contributions from the internal order, the value of the bare surface tension σ will have to be adjusted to keep σ_R zero. In what follows, we will ignore $\mathcal{H}_\sigma$ with the understanding that it is really present if we want to keep track of how σ_R actually becomes zero.

The hexatic free energy is the contribution to the energy from fluctuations in the local bond order parameter. Since the hexatic order parameter **m** has a fixed magnitude and there are no external fields aligning **m** along a particular direction, the lowest nontrivial contribution to the energy associated with **m** arises from its gradients,

$$\mathcal{H}_A=\frac{1}{2}K_A\int d^2u\sqrt{g}g^{\alpha\beta}D_\alpha\mathbf{m}\cdot D_\beta\mathbf{m}, \tag{9}$$

where D_α is a covariant derivative since the bond order parameter is frustrated by the rotation of tangent vectors that occurs under parallel transport on a curved surface. The amount of frustration is given by the gauge field A_α, i.e., the covariant derivative of $\mathbf{e}_a$ in direction α defines the gauge field A_α. Under parallel transport in direction du^α, each $\mathbf{e}_a$ is rotated by an angle $A_\alpha du^\alpha$. Thus the gauge field A_α is defined by

$$D_\alpha\mathbf{e}_a=-A_\alpha\varepsilon_{ab}\mathbf{e}_b, \tag{10}$$

where ε_{ab} is the antisymmetric tensor with $\varepsilon_{12}=-\varepsilon_{21}=1$ and $A_\alpha\varepsilon_{ab}$ is called the spin connection and describes how the basis vector $\mathbf{e}_a$ rotates under parallel transport according to the Gaussian curvature K of the surface. In fact, A_α is related to K. The curl of the gauge field A_α is the Gaussian curvature

$$\gamma^{\alpha\beta}D_\alpha A_\beta=K, \tag{11}$$

where $\gamma^{\alpha\beta}$ is the antisymmetric tensor defined via

$$\gamma_{\alpha\beta}=\mathbf{N}\cdot(\mathbf{t}_\alpha\times\mathbf{t}_\beta)=\sqrt{g}\varepsilon_{\alpha\beta},\quad \gamma^{\alpha\beta}=g^{\alpha\alpha'}g^{\beta\beta'}\gamma_{\alpha'\beta'}, \tag{12}$$

In terms of the local bond angle $\Theta(\Omega)$, the covariant derivative of **m** writes

$$D_\alpha\mathbf{m}=(D_\alpha m_a)\mathbf{e}_a+m_a(D_\alpha\mathbf{e}_a)=(D_\alpha m_a)\mathbf{e}_a-m_aA_\alpha\varepsilon_{ab}\mathbf{e}_b$$
$$=(D_\alpha\Theta)(-\sin\Theta\,\mathbf{e}_1+\cos\Theta\,\mathbf{e}_2)-A_\alpha(\cos\Theta\,\mathbf{e}_2-\sin\Theta\,\mathbf{e}_1)$$
$$=(D_\alpha\Theta-A_\alpha)\mathbf{m}_\perp, \tag{13}$$

where $\mathbf{m}_\perp=-\sin\Theta\mathbf{e}_1+\cos\Theta\mathbf{e}_2$, satisfying $\mathbf{m}\cdot\mathbf{m}_\perp=0$. Then the hexatic energy writes

$$\mathcal{H}_A=\frac{1}{2}K_A\int d^2u\sqrt{g}g^{\alpha\beta}(\partial_\alpha\Theta-A_\alpha)(\partial_\beta\Theta-A_\beta). \tag{14}$$

Thus we have the Hamiltonian $\mathcal{H}=\mathcal{H}_\kappa+\mathcal{H}_{\rm A}$ to describe fluctuating hexatic membranes.

To gain some physical understanding of a spherical hexatic membrane, we examine the ground states. Since we are interested in the limit $\beta\kappa\gg1$ and in this limit $\mathcal{H}_\kappa$ dominates, we first minimize $\mathcal{H}_\kappa$ over the shape fluctuation field ρ, which gives $\rho(\Omega)=0$, and then we minimize $\mathcal{H}_A$ over Θ with $\rho(\Omega)=0$ and find

$$\left.\frac{\delta\mathcal{H}_A^0}{\delta\Theta(\Omega)}\right|_{\Theta=\Theta^0}=\frac{1}{\sqrt{g^0}}\partial_b g^{0ab}(\partial_a\Theta^0-A_a^0)=0, \tag{15}$$

where the superscript 0 stands for the rigid sphere with $\rho(\Omega)=0$. In Ref. [10], Lubensky and Prost show that in the ground state 12 disclinations of strength $2\pi/6$ are arranged at the vertices of icosahedron inscribed in the sphere. A disclination at $\mathbf{u}=\mathbf{u}_i$ with strength q_i gives rise to a singular contribution $\Theta_0^{\rm sing}$ to Θ^0 satisfying

$$\oint_\Gamma du^\alpha\partial_\alpha\Theta_0^{\rm sing}=q_i, \tag{16}$$

where Γ is a contour enclosing $\mathbf{u}_i$. Thus, in general, $\partial_\alpha\Theta^0=\partial_\alpha\Theta_0'+v_\alpha^0$, where Θ_0' is nonsingular, $v_\alpha^0=\partial_\alpha\Theta_0^{\rm sing}$, and

$$\gamma^{\alpha\beta}D_\alpha v_\beta^0=n^0(\Omega), \tag{17}$$

where

$$n^0(\Omega)=\frac{2\pi}{6}\sum_{i=1}^{12}\delta(\Omega-\Omega_i), \tag{18}$$

which is the disclination density in the ground state and Ω_i's are the coordinates of the vertices of icosahedron. Since $\partial_\alpha\Theta^0-A_\alpha^0$ satisfies $D^\alpha(\partial_\alpha\Theta^0-A_\alpha^0)=0$, it is divergence-free and purely transverse. Accordingly $\partial_\alpha\Theta^0-A_\alpha^0$ can be written in terms of the curl of scalar fields and by applying the operator $\gamma^{\beta\alpha}D_\beta$ to $\partial_\alpha\Theta^0-A^0_\alpha$ we find these scalar fields to be related to the Gaussian curvature K_0 of the rigid sphere and the ground-state disclination density on the rigid sphere,

$$\gamma^{\beta\alpha}D_\beta(\partial_\alpha\Theta^0-A^0_\alpha)=\gamma^{\beta\alpha}D_\beta v^0_\alpha-\gamma^{\beta\alpha}D_\beta A^0_\alpha=n^0-K_0, \tag{19}$$

where K_0 is a Gaussian curvature of the rigid sphere and n^0 is the disclination density in the ground state.

Now taking into account the bond angle fluctuations around Θ^0 and the shape fluctuations around the sphere,

$$\Theta=\Theta^0+\widetilde{\Theta},\quad A_\alpha=A^0_\alpha+\delta A_\alpha, \tag{20}$$

the full Hamiltonian writes $\mathcal{H}=\mathcal{H}_0+\delta\mathcal{H}$,

$$\mathcal{H}_0=\frac{1}{2}K_A\int d\Omega(\partial^\alpha\Theta^0-A^{0\alpha})(\partial_\alpha\Theta^0-A^0_\alpha),$$

$$\delta\mathcal{H}=\frac{1}{2}\kappa\int d\Omega[(\nabla^2+2)\rho]^2+\frac{1}{2}K_A\int d\Omega(\partial^\alpha\widetilde{\Theta}-\delta A^\alpha)\times(\partial_\alpha\widetilde{\Theta}-\delta A_\alpha)+O(\rho^3). \tag{21}$$

The angle fluctuation field $\widetilde{\Theta}(\Omega)$ can also have disclinations of strength $q=2\pi(k/6)$, where k is an integer, due to the thermal fluctuation [11]. Thus $\partial_\alpha\widetilde{\Theta}$ can be decomposed into singular and nonsingular parts $\partial_\alpha\widetilde{\Theta}=\partial_\alpha\Theta^\parallel+v_\alpha$, where $\Theta^\parallel$ is nonsingular, $v_\alpha=\partial_\alpha\widetilde{\Theta}^{\rm sing}$, and

$$\gamma^{\alpha\beta}D_\alpha v_\beta=n(\Omega),\quad n(\Omega)=\sum_i q_i\delta(\Omega-\Omega_i), \tag{22}$$

where $n(\Omega)$ is the thermally excited disclination density with disclinations of strength q_i at Ω_i. The vector v_α can always be chosen so that it is purely transverse, i.e., $D_\alpha v^\alpha=0$. In the hexatic Hamiltonian, $\partial_\alpha\widetilde{\Theta}$ always occurs in the combination $\partial_\alpha\widetilde{\Theta}-\delta A_\alpha$. The spin connection δA_α can and will, in general, have both a longitudinal and a transverse component. However, one can always redefine $\Theta^\parallel$ to include the longitudinal part of δA_α. This amounts to choosing locally rotated orthonormal vectors $\mathbf{e}_1(\mathbf{u})$ and $\mathbf{e}_2(\mathbf{u})$ so that $D_\alpha\delta A^\alpha=0$. Thus we may take both v_α and δA_α to be transverse and the hexatic Hamiltonian

$$\frac{1}{2}K_A\int d\Omega(\partial^\alpha\Theta^\parallel+v^\alpha-\delta A^\alpha)(\partial_\alpha\Theta^\parallel+v_\alpha-\delta A_\alpha)=\mathcal{H}_\parallel+\mathcal{H}_\perp \tag{23}$$

can be decomposed into a regular longitudinal part

$$\mathcal{H}_\parallel=\frac{1}{2}K_A\int d\Omega\,\partial^\alpha\Theta^\parallel\partial_\alpha\Theta^\parallel \tag{24}$$

and a transverse part

$$\mathcal{H}_\perp=\frac{1}{2}K_A\int d\Omega(v^\alpha-\delta A^\alpha)(v_\alpha-\delta A_\alpha), \tag{25}$$

where the cross term $\int d\Omega(v_\alpha-\delta A_\alpha)\partial^\alpha\Theta^\parallel$ is dropped since $D^\alpha(v_\alpha-A_\alpha)=0$.

It costs energy $\epsilon_c(k)$ to create the core of a disclination of strength k. (We assume, for the moment, that the core energies of the positive and negative disclinations are the same. See, however, Refs. [1,12], [13].) Thus partition sums should be weighted by a factor $y_k=e^{-\beta\epsilon_c(k)}$ for each disclination of strength k. Since $\epsilon_c(k)\sim k^2$, we may, at low temperature, restrict our attention to configurations in which only configurations of strength ±1 appear. Let $N_\pm$ be the number of disclinations of strength ±1 and let $\mathbf{u}_{i^\pm}$ be the coordinate of the core of the disclination with strength ±1 labeled by i. The hexatic membrane partition function can then be written as

$$\mathcal{Z}(\kappa,K_A,y)=\mathrm{Tr}_v y^N\int\mathcal{D}\mathbf{R}\int\mathcal{D}\Theta^\parallel e^{-\beta\mathcal{H}_\kappa}e^{-\beta(\mathcal{H}_\parallel+\mathcal{H}_\perp)}, \tag{26}$$

where $y=y_1$, and $N=N_++N_-$. $\mathcal{H}_\perp$ depends on all of the disclination coordinates $\Omega_{\nu^\pm}$ where $\nu^\pm=1,2,\ldots,N_\pm$, and Tr_v is the sum over all possible disclination distribution with the topological constraint [14]

$$\mathrm{Tr}_v=\sum_{N_+,N_-}\frac{\delta_{N_+,N_-}}{N_+!N_-!}\prod_{\nu^+}\int\frac{d\Omega_{\nu^+}}{a^2}\prod_{\nu^-}\int\frac{d\Omega_{\nu^-}}{a^2}, \tag{27}$$

where a^2 is a molecular solid angle. The Kronecker factor δ_{N_+,N_-} in Tr_v imposes the topological constraint that the total disclination strength on a sphere is 2 since with $N_+=N_-$ we have 12 ground-state disclinations with the strength 1/6 giving the total disclination strength $12\times(1/6)=2$.

The hexatic model of Eq. (26) can easily be converted to a Coulomb gas model using

$$\gamma^{\alpha\beta}D_\alpha(v_\beta-\delta A_\beta)=n-\delta K\equiv\mathcal{C}, \tag{28}$$

which follows from Eqs. (11) and (22) where δK is the deviation of the Gaussian curvature from the rigid sphere. The quantity $\mathcal{C}=n-\delta K$ is a ''charge'' density with contributions arising both from disclinations and from Gaussian curvature. Equation (28) implies

$$v_\alpha-\delta A_\alpha=-\gamma_\alpha{}^\beta D_\beta\frac{1}{\Delta}\mathcal{C}, \tag{29}$$

where we used $\gamma_{\alpha\lambda}D^\lambda\gamma^{\alpha\beta}D_\alpha=-\Delta$ and $\Delta=D^\alpha D_\alpha=(1/\sqrt{g})\partial_\alpha\sqrt{g}g^{\alpha\beta}\partial_\beta$ is the Laplacian on a surface with metric tensor $g_{\alpha\beta}$ acting on a scalar. Recall [Eq. (12)] that $\gamma_\alpha{}^\beta$ rotates a vector by $\pi/2$ so that $v_\alpha-\delta A_\alpha$ is perpendicular to $D_\beta(-\Delta)^{-1}\mathcal{C}$ and is thus manifestly transverse. Using Eq. (29) in Eq. (25), we obtain

$$\mathcal{Z}=\mathrm{Tr}_v y^N\int\mathcal{D}\mathbf{R}\int\mathcal{D}\Theta^\parallel e^{-\beta\mathcal{H}_\kappa-\beta\mathcal{H}_\parallel-\beta\mathcal{H}_c}, \tag{30}$$

where

$$\mathcal{H}_c=\frac{1}{2}K_A\int d\Omega\,\frac{\gamma_\alpha^\beta D_\beta}{\Delta}\mathcal{C}(\Omega)\frac{\gamma^{\alpha\lambda}D_\lambda}{\Delta}\mathcal{C}(\Omega)$$
$$=\frac{1}{2}K_A\int d\Omega\; d\Omega'\mathcal{C}(\Omega)$$
$$\times\left(\frac{\gamma_\alpha^\beta D_\beta\gamma^{\alpha\lambda}D_\lambda}{\Delta^2}\delta(\Omega-\Omega')\right)\mathcal{C}(\Omega')$$
$$=\frac{1}{2}K_A\int d\Omega\; d\Omega'\mathcal{C}(\Omega)\left(-\frac{1}{\Delta}\delta(\Omega-\Omega')\right)\mathcal{C}(\Omega') \quad (31)$$

is the Coulomb Hamiltonian associated with the charge $\mathcal{C}$. Since the longitudinal variable $\Theta^\parallel$ appears only quadratically in $\mathcal{H}_\parallel$, the trace over $\Theta^\parallel$ can be done directly giving the Liouville action [15] arising from the conformal anomaly

$$\int \mathcal{D}\Theta^\parallel e^{-\beta\mathcal{H}_\parallel}=e^{-\beta\mathcal{H}_L}, \quad (32)$$

where

$$\beta\mathcal{H}_L=\frac{1}{8\pi a^2}\int d\Omega-\frac{1}{24\pi}\int d\Omega\; d\Omega' K(\Omega)$$
$$\times\left(-\frac{1}{\Delta}\delta(\Omega-\Omega')\right)K(\Omega'). \quad (33)$$

The Coulomb gas partition function can thus be written

$$\mathcal{Z}=\mathrm{Tr}_v y^N\int \mathcal{D}\mathbf{R}e^{-\beta\mathcal{H}_\kappa-\beta\mathcal{H}_L-\beta\mathcal{H}_C}. \quad (34)$$

The Coulomb gas model can be converted, by following standard procedures, into a sine-Gordon model. The first step is to carry out a Hubbard-Stratonovich transformation on $\beta\mathcal{H}_C$,

$$e^{-\beta\mathcal{H}_C}=e^{\beta\mathcal{H}_L}\int \mathcal{D}\Phi\exp\left(-\frac{1}{2}(\beta K_A)^{-1}\int d\Omega\,\partial^\alpha\Phi\,\partial_\alpha\Phi\right)$$
$$\times\exp\left(i\int d\Omega\mathcal{C}\Phi\right), \quad (35)$$

where the Liouville factor $e^{\beta\mathcal{H}_L}$ is needed to ensure that $e^{-\beta\mathcal{H}_C}$ is one when $\mathcal{C}=0$. Inserting this in Eq. (34), we obtain

$$\mathcal{Z}=\mathrm{Tr}_v y^N\int \mathcal{D}\mathbf{R}\;\mathcal{D}\Phi e^{-\beta\mathcal{H}_\kappa-\beta\mathcal{H}_\Phi}\exp\left(i\int d\Omega(n-\delta K)\Phi\right), \quad (36)$$

where

$$\beta\mathcal{H}_\Phi=\frac{1}{2}(\beta K_A)^{-1}\int d\Omega\;\partial^\alpha\Phi\,\partial_\alpha\Phi. \quad (37)$$

The only dependence on disclinations is now in the term linear in n. Thus, to carry out Tr_v, we need only to evaluate

$$\mathrm{Tr}_v y^N\exp\left(i\int d\Omega\; n\Phi\right)=\sum_{N_+,N_-}\frac{1}{N_+!N_-!}\delta_{N_+,N_-}y^{N_++N_-}\left(\int\frac{d\Omega}{a^2}e^{2\pi i\Phi(\Omega)/6}\right)^{N_+}\left(\int\frac{d\Omega}{a^2}e^{-2\pi i\Phi(\Omega)/6}\right)^{N_-}$$
$$=\sum_{N_+,N_-}\frac{1}{N_+!N_-!}\int\frac{d\omega}{2\pi}\left(y\int\frac{d\Omega}{a^2}e^{i\{2\pi[\Phi(\Omega)/6]-\omega\}}\right)^{N_+}\left(y\int\frac{d\Omega}{a^2}e^{-i\{2\pi[\Phi(\Omega)/6]-\omega\}}\right)^{N_-}$$
$$=\int\frac{d\omega}{2\pi}\exp\left((2y/a^2)\int d\Omega\cos[2\pi(\Phi/6)-\omega]\right). \quad (38)$$

Thus

$$\mathcal{Z}=\int\frac{d\omega}{2\pi}\int \mathcal{D}\Phi\int \mathcal{D}\mathbf{R}\;e^{-\beta\mathcal{H}_\kappa}e^{-\beta\mathcal{H}_\Phi}\exp\left((2y/a^2)\right.$$
$$\left.\times\int d\Omega\cos[2\pi(\Phi/6)-\omega]\right)\exp\left(-i\int d\Omega\,\Phi\,\delta K\right). \quad (39)$$

We can now change variables, setting $\Phi=(6/2\pi)(\Phi'+\omega)$. The term linear in the Gaussian curvature then becomes

$$-i\int d\Omega\,\delta K\frac{6}{2\pi}(\omega+\Phi')=-i\frac{p}{2\pi}\int d\Omega\;\Phi'\,\delta K, \quad (40)$$

where we used $\int d\Omega\,\delta K=0$. The integral over ω in Eq. (39) is now trivial, and dropping the prime we obtain

$$\mathcal{Z}=\int \mathcal{D}\Phi\int \mathcal{D}\mathbf{R}e^{-\beta\mathcal{H}_\kappa}e^{-\mathcal{L}}, \quad (41)$$

where

$$\mathcal{L}=\frac{1}{2}(\beta K_A)^{-1}\left(\frac{6}{2\pi}\right)^2\int d\Omega\,\partial^\alpha\Phi\,\partial_\alpha\Phi-\frac{2y}{a^2}\int d\Omega\cos\Phi$$
$$-i\frac{6}{2\pi}\int d\Omega\;\Phi\,\delta K \quad (42)$$

is the sine-Gordon action on a fluctuating surface of genus zero. The first two terms of this action are the gradient and cosine energies present on a rigid sphere. The final term provides the principal coupling between Φ and fluctuations in the metric. It is analogous to the dilaton coupling [16] of string theory, though here the coupling constant is imaginary rather than real. Note that the Liouville action is not explicitly present in Eq. (41).

In the regime $\beta\kappa \gg 1$, we can truncate the higher-order terms in ρ. In the normal gauge, the partition function becomes

$$\mathcal{Z} = \int \mathcal{D}\rho \mathcal{D}\Phi \exp\left[-\frac{1}{2}\beta\kappa \int d\Omega [(\nabla^2+2)\rho]^2 - \frac{1}{2}\beta\Gamma \int d\Omega (\nabla\Phi)^2 + \frac{2y}{a^2}\int d\Omega \cos\Phi + i\frac{6}{2\pi}\int d\Omega \Phi(\nabla^2+2)\rho \right], \tag{43}$$

where $\beta\Gamma \equiv 36/4\pi^2 \beta K_A$ and we used $\delta K = (\nabla^2+2)\rho$. To lowest order in ρ, the shape fluctuation field ρ is linearly coupled to the scalar field Φ, which is the conjugate field to the disclinations. In Ref. [1], we have shown that similar coupling in the fluctuating flat membrane is quadratic in the shape fluctuation field.

Integrating over the shape fluctuation field ρ gives the effective Hamiltonian for the conjugate field to the disclinations

$$\mathcal{Z} = \int \mathcal{D}\Phi \exp\left[-\frac{1}{2}\beta\Gamma \int d\Omega [(\nabla\Phi)^2 + \mu^2\Phi^2] + \frac{2y}{a^2}\int d\Omega \cos\Phi \right], \tag{44}$$

with $\mu^2 = K_A/\kappa$. This is the massive sine-Gordon theory. The shape fluctuations induce the mass term for Φ field and screen the Coulombic interaction between the disclinations giving the Yukawa interaction between them. This partition function is equivalent to that of the Yukawa gas Hamiltonian on the rigid sphere with radius R,

$$\mathcal{H}_{\text{Yukawa}} = \frac{1}{2}\beta K_A \int d\Omega\, n(\Omega) \frac{1}{-\nabla^2+\mu^2} n(\Omega) = \frac{1}{2}\beta K_A \sum_{i,j} q_i q_j G(\Omega_i - \Omega_j), \tag{45}$$

where

$$G(\Omega_i - \Omega_j) = \sum_l \frac{2l+1}{l(l+1)+\mu^2} P_l(\cos\omega_{ij}) = -\frac{\pi}{\cos\left[\left(\nu+\frac{1}{2}\right)\pi\right]} P_\nu(-\cos\omega_{ij}), \tag{46}$$

where $P_\nu(\,)$ is the Legendre polynomial with degree $\nu = -1/2 \pm (\sqrt{1-4\mu^2})/2$ and ω_{ij} is the angle between two disclinations at Ω_i and Ω_j. For $0 \leq \mu \leq 1/2$, the degree of the Legendre polynomial ν is real and the length scale introduced by $\lambda_d \equiv (\mu/R)^{-1}$ is larger than the system size $2R$ after recovering the original length scale by mapping the unit sphere back to the sphere with the radius R. On the other hand, if $\mu > 1/2$, ν is a complex number, $\nu = -1/2 \pm i\tau$, where $\tau = (\sqrt{4\mu^2-1})/2$ and $\lambda_d < 2R$. The length scale introduced by $\lambda_d = R/\mu$ may be interpreted as the Debye screening length arising from shape fluctuations.

The interaction energy between two disclinations i and j at positions Ω_i and Ω_j with strength q_i and q_j is given by $q_i q_j G(d_{ij})$, where $d_{ij} = 2R\sin(\omega_{ij}/2)$ is the chordal distance between two disclinations on the sphere with radius R. The interaction $G(d_{ij})$ has the limiting forms

$$G(d_{ij}) \simeq \begin{cases} -\frac{1}{2}\ln(d_{ij}/2), & \mu^{-1} \gg 2, \quad d_{ij} \ll 2R \\ -\frac{1}{2}\ln(d_{ij}/2), & \mu^{-1} \ll 2, \quad d_{ij} \ll \lambda_d \\ e^{-d_{ij}/\lambda_d}, & \mu^{-1} \ll 2, \quad d_{ij} \gg \lambda_d . \end{cases} \tag{47}$$

Following the analogy of the two-dimensional Coulomb gas, when the screening length is much larger than the system size $\lambda_d \gg 2R$, the induced mass term arising from shape fluctuations is irrelavent for the KT transition and the KT transition temperature is given by $T_c = \pi K_A/72$ for $\mu \ll 1/2$. However, for $\mu \gg 1/2$, the screening length is shorter than the system size $\lambda_d \ll 2R$ and the mass term is relavent for the KT transition and changes the universality class of the system. There is no KT transition at nonzero temperature. The disclinations are always unbound at nonzero temperature and the KT transition temperature vanishes. Thus we find the crossover at $\mu = 1/2$.

In conclusion, we present the effect of shape fluctuations on the interaction of the disclinations on a spherical surface with genus zero. We have confirmed that the screened interaction is of the same form as the vortex line interactions in type-II superconductors. In these superconductors, screening of vortex line interactions drives the Kosterlitz-Thouless transition temperature to zero for an infinite superconductor in zero magnetic field [17]. Likewise, the screening of the disclination interaction on the fluctuating spherical surface drives the KT transition temperature to zero for $\mu^{-1} \ll 2$ in which the screening length is shorter than the system size. However, when $\mu^{-1} \gg 2$, the effect of shape fluctuations is irrelavent and the effective KT transition occurs at a finite temperature.

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